

A note on the number of edges of the Jaco Graph, $J_n(1), n \in \mathbb{N}$

(Johan Kok, Vivian Mukungunugwa)¹

Abstract

Kok et.al. [3] introduced Jaco Graphs (*order 1*). It is hoped that as a special case, a closed formula can be found for the number of edges of a finite Jaco Graph $J_n(1)$. However, the algorithms discussed in Ahlback et al.[1] suggest this might not be possible. Finding a closed formula for the number of edges of a Jaco Graph, $J_n(1), n \in \mathbb{N}$ remains an interesting open problem. In this note we present three alternative, *formula*.

Keywords: Jaco graph, Directed graph, Hope graph, Jaconian vertex, Number of edges.

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1 Introduction

The infinite Jaco graph (*order 1*) was introduced in [3], and defined by $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$. The graph has four fundamental properties which are; $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if v_k , for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

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2 Number of edges of a Jaco Graph, $J_n(1), n \in \mathbb{N}$

It is hoped that a closed formula can be found for the number of edges of a finite Jaco Graph $J_n(1)$. However, the algorithms discussed in Ahlback et al.[1] suggest this might not be possible. Finding a closed formula for the number of edges of a Jaco Graph, $J_n(1), n \in \mathbb{N}$ remains an interesting open problem. We present three alternative, *formula*.

Lemma 2.1. $\epsilon(J_n(1)) = \epsilon(J_{n-1}(1)) + d^-(v_n)$.

Proof. Trivial. □

We now present the adapted Fisher Algorithm. The original Fisher Algorithm is found in [3].

2.1 The adapted Fisher Algorithm for $\{J_i(1), i \in \{4, 5, 6, \dots, s \in \mathbb{N}\}$

The family of finite Jaco Graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

Note that rows 1, 2 and 3 follow easily from the definition.

Step 0: Set $j = 4$, then set $i = j$ and $s \geq 4$.

Step 1: Set $ent_{1i} = i$.

Step 2: Set $ent_{2i} = ent_{1(i-1)} - ent_{4(i-1)}$. (Note that $d^-(v_i) = v(\mathbb{H}_{i-1}(1)) = (i-1) - \Delta(J_{i-1}(1))$).

Step 3: Set $ent_{3i} = ent_{1i} - ent_{2i}$. (Note that $d^+(v_i) = i - d^-(v_i)$).

Step 4: Set $ent_{5i} = ent_{5(i-1)} + ent_{2i}$. (Lemma 2.1)

Step 5: Set $j = i + 1$, then set $i = j$. If $i \leq s$, go to Step 1, else go to Step 6.

Step 6: Exit.

First recursive formula: Note that Lemma 2.1 provides the *first* recursive formula to determine the number of edges of $J_n(1)$. Using the adapted Fisher Algorithm together with Lemma 2.1 the table below follows easily.

$\phi(v_i) \rightarrow i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$\Delta(J_i(1))$	$\epsilon(J_i(1)) = \epsilon(J_{i-1}(1)) + d^-(v_i)$
1= f_2	0	1	0	0
2= f_3	1	1	1	1
3= f_4	1	2	2	2
4	1	3	2	3
5= f_5	2	3	3	5
6	2	4	3	7
7	3	4	4	10
8= f_6	3	5	5	13
9	3	6	5	16
10	4	6	6	20
11	4	7	7	24
12	4	8	7	28
13= f_7	5	8	8	33
14	5	9	8	38
15	6	9	9	44
16	6	10	10	50
17	6	11	10	56
18	7	11	11	63
19	7	12	11	70
20	8	12	12	78
21= f_8	8	13	13	86
22	8	14	13	94
23	9	14	14	103
24	9	15	15	112
25	9	16	15	121
26	10	16	16	131
27	10	17	16	141
28	11	17	17	152
29	11	18	18	163
30	11	19	18	174
31	12	19	19	186
32	12	20	20	198
33	12	21	20	210
34= f_9	13	21	21	223
35	13	22	21	236

Second formula: It is well known that $\epsilon(J_n(1)) = \sum_{i=1}^n d^-(v_i)$. Since $d^-(v_n) = n - d^+(v_n)$, the number of edges is also given by $\epsilon(J_n(1)) = \frac{1}{2}n(n-1) - \sum_{i=1}^n d^+(v_i)$. Furthermore, for $n \geq 2$ we have $d^+(v_1) = 1$, so we rather consider $\epsilon(J_n(1)) = (\frac{1}{2}(n(n-1) - 1) - \sum_{i=2}^n d^+(v_i))$. Bettina's Theorem [4] provides for a method to determine $d^+(v_i), \forall i \in \mathbb{N}$.

Example 1. For the Jaco Graph $J_{15}(1)$ we have;

$$\epsilon(J_{15}(1)) = \frac{1}{2}.15.(15+1) - 1 - \sum_{i=2}^{15} d^+(v_i) = 119 - \sum_{i=2}^{15} d^+(v_i).$$

Now, $1 = f_2, 2 = f_3, 3 = f_4, 4 = f_4 + f_2, 5 = f_5, 6 = f_5 + f_2, 7 = f_5 + f_3, 8 = f_6, 9 = f_6 + f_2, 10 = f_6 + f_3, 11 = f_6 + f_4, 12 = f_6 + f_4 + f_2, 13 = f_7, 14 = f_7 + f_2$, and $15 = f_7 + f_3$.

From Bettina's Theorem it follows that;

$$\sum_{i=2}^{15} d^+(v_i) = f_2 + f_3 + (f_3 + f_1) + f_4 + (f_4 + f_1) + (f_4 + f_2) + f_5 + (f_5 + f_1) + (f_5 + f_2) + (f_5 + f_3) + (f_5 + f_3 + f_1) + f_6 + (f_6 + f_1) + (f_6 + f_2) = 5f_1 + 4f_2 + 4f_3 + 3f_4 + 5f_5 + 3f_6 = f_1 + 5f_3 + 5f_5 + 3f_7 = 75.$$

So, $\epsilon(J_{15}(1)) = 119 - 75 = 44$.

Third formula: The third formula follows from Proposition 2.2 and 2.3.

Proposition 2.2. *If for the finite Jaco Graph $J_n(1), n \in \mathbb{N}$, n can be expressed as $n = m + d^+(v_m) + d^+(v_m)$ we have that:*

$$\epsilon(J_n(1)) = \frac{1}{2} \left(\sum_{i=1}^m i + \sum_{i=0}^{j_{max}} (d^+(v_{m-i}) - i)_{d^+(v_{m-j_{max}}) - j_{max} \geq 1} + d^+(v_m)(d^+(v_m) - 1) \right).$$

Proof. Consider the the Jaco Graph $J_n(1)$ with $n = m + d^+(v_m)$. Now consider the Jaco Graph $J_m(1)$. From the definition, $J_m(1)$ was obtained by lobbing off the vertices $v_{m+1}, v_{m+2}, \dots, v_{m+d^+(v_m)}$ together with all arcs (edges) incident to the said vertices. Reconstructing $J_n(1)$ can follow in three steps.

Step 1: Link the arcs $(v_m, v_{m+1}), (v_m, v_{m+2}), \dots, (v_m, v_{m+d^+(v_m)})$ to construct the graph $J^*(1)$. After these additional $d^+(v_m)$ arcs are added we have $d(v_m) = m$ in $J^*(1)$ as required by definition. Note that $d^+(v_m)$ will be denoted $(d^+(v_m) - 0)$.

Step 2: Because vertex v_m is the Jaconian vertex of $J_n(1)$, we link all arcs (edges) to vertices

$v_{m+1}, v_{m+2}, \dots, v_{m+d^+(v_m)}$ to ensure the Hope graph of $J_n(1)$ is constructed (See [3].) Hence $d^+(v_m)(d^+(v_m) - 1)$ arcs are added.

Step 3: If $d(v_{m-1}) < (m - 1)$ then exactly $(d^+(v_{m-1}) - 1)$ *bridging arcs* are needed. So link $(v_{m-1}, v_{m+1}), (v_{m-1}, v_{m+2}), (v_{m-1}, v_{m+3}), \dots, (v_{m-1}, v_{m+d^+(v_{m-1})-1})$. Recursively we add *bridging arcs* for vertices $v_{m-2}, v_{m-3}, \dots, v_{m-j_{max}}$ such that $(d^+(v_{m-j_{max}}) - j_{max}) \geq 1$.

Now the original Jaco graph $J_n(1)$ has been reconstructed. From the well-known general result, $\epsilon(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v)$, the result:

$$\epsilon(J_n(1)) = \frac{1}{2} \left(\sum_{i=1}^m i + \sum_{i=0}^{j_{max}} (d^+(v_{m-i}) - i)_{d^+(v_{m-j_{max}}) - j_{max} \geq 1} + d^+(v_m)(d^+(v_m) - 1) \right), \text{ follows. } \square$$

Example 2. Determine $\epsilon(J_{31}(1))$. Now, $31 = 19 + d^+(v_{19}) = 19 + 12$. So we have:

$$\begin{aligned} \epsilon(J_{31}(1)) &= \frac{1}{2} \left(\sum_{i=1}^{19} i + \sum_{i=0}^{j_{max}} (d^+(v_{19-i}) - i) + d^+(v_{19})(d^+(v_{19}) - 1) \right) \\ &= \frac{1}{2} \left(\frac{1}{2}(20 \cdot 19) + [(12 - 0) + (11 - 1) + (11 - 2) + (10 - 3) + (9 - 4) + (9 - 5) + (8 - 6) + (8 - 7)]_{=50} + [12 \cdot 11]_{=132} \right) \\ &= \frac{1}{2}(190 + 50 + 132) = 186. \end{aligned}$$

Note that not all $n \in \mathbb{N}$ can uniquely be written as $n = m + d^+(v_m)$ for $m \in \mathbb{N}$. From Corollary 3.5 in [4] the next proposition follows.

Proposition 2.3. *If, for the Jaco Graph $J_{n-1}(1)$ the integer $(n - 1)$ cannot be expressed as $n - 1 = m_{=d^+(v_{n-1})} + d^+(v_m)$, then $n = m_{=d^+(v_{n-1})} + d^+(v_m) = m_{=d^+(v_n)} + d^+(v_m)$, and $\epsilon(J_{n-1}(1)) = \epsilon(J_n(1)) - d^+(v_m)$.*

Proof. The result follows from the reverse of Lemma 2.1 because $d^-(v_n) = d^+(v_m)$. \square

Example 3. Determine $\epsilon(J_{17}(1))$. Note that $d^+(v_{17}) + d^+(v_{d^+(v_{17})}) = 18 \neq 17$. However, $d^+(v_{18}) + d^+(v_{d^+(v_{18})}) = 18$, so $\epsilon(J_{17}(1)) = \epsilon(J_{18}(1)) - d^+(v_{11})$. Hence $\epsilon(J_{17}(1)) = 63 - 7 = 56$.

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